Linear Algebra
[KOMS120301] - 2023/2024

9.1 - Vectors in Space

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Learning objectives

After this lecture, you should be able to:

1. explain the concept of Euclidean space (n-space);
2. perform operations on vectors such as addition and multiplication;
3. explain the geometric interpretation of linear combination of vectors;
4. explain the concept of linear independence of vectors;
5. implement properties of vectors operations in $\mathbb{R}^n$ to problem solving.

Part 1: Vector Space

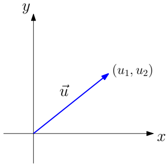
What is an n-space?

Recall our previous discussion...

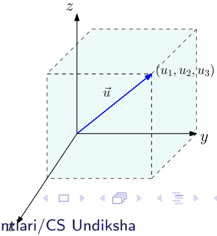
- An ordered n-tuple is a sequence of *real numbers*: $(a_1, a_2, \dots, a_n)$ (or, can be seen as a vector).
- An n-space is a set of all n-tuples of real numbers. Usually denoted as $\mathbb{R}^n$. For $n = 1$, $\mathbb{R}^1 \equiv \mathbb{R}$.
  - This space is where vectors are defined
- The n-space $\mathbb{R}^n$ is also called Euclidean space.

Example:

Vector in $\mathbb{R}^2$



Vector in $\mathbb{R}^3$



Vectors in n-space

- An n-tuple in $\mathbb{R}^n$, e.g. $u = (u_1, u_2, \ldots, u_n)$ is called a point or a vector.
- The numbers u_i are called coordinates, components, entries, or elements of u.
- When referring to $\mathbb{R}^n$, an element of $\mathbb{R}$ is called scalar.
- The vector $(0, 0, \ldots, 0)$ is called zero vector.
  - Example: the zero vector in $\mathbb{R}^2$ is $(0, 0)$, and the zero vector in $\mathbb{R}^3$ is $(0, 0, 0)$
- Vectors $\mathbf{u}$ and $\mathbf{v}$ are equal if they have the same number of components, and the corresponding components are equal.

Row vectors and column vectors

A vector in $\mathbb{R}^n$ can be written horizontally (this is called row vector) or vertically (called column vector).

$$u = [a_1, a_2, \ldots, a_n] \qquad\qquad u = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_3 \end{bmatrix}$$

Note: any operation defined for row vectors is defined analogously for column vectors. From now on, vectors are often written as row vectors.

Part 2: Vectors Operations

Vectors addition and scalar multiplication

Let u and v be vectors in $\mathbb{R}^n$, say:

$$u = (a_1, a_2, \ldots, a_n) \quad \text{and} \quad v = (b_1, b_2, \ldots, b_n)$$

The sum $u + v$ is defined as:

$$u + v = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n)$$

If $k \in \mathbb{R}$, the scalar product or product ku is defined as:

$$ku = k(a_1, a_2, \ldots, a_n) = (ka_1, ka_2, \ldots, ka_n)$$

The negative and subtraction (the difference of u and v) are defined as:

$$-u = (-1)u \quad \text{and} \quad u - v = u + (-v)$$

Note: $u + v,\ ku,\ -u,\ u - v$ are also vectors in $\mathbb{R}^n$.

The zero vector and one vector

The *zero vector* $0 = (0, 0, \ldots, 0)$ and the *one vector* $1 = (1, 1, \ldots, 1)$ in $\mathbb{R}^n$ are similar to the scalar 0 and 1 in $\mathbb{R}$.

- For a vector $u = (a_1, a_2, \ldots, a_n)$, then:

$$u + 0 = (a_1 + 0, a_2 + 0, \ldots, a_n + 0) = (a_1, a_2, \ldots, a_n) = u$$
$$1u = 1(a_1, a_2, \ldots, a_n) = (a_1, a_2, \ldots, a_n) = u$$

Part 3: Linear Combination of Vectors

Linear combination

Given vectors $u_1, u_2, \ldots, u_n \in \mathbb{R}^n$ and scalars $k_1, k_2, \ldots, k_n \in \mathbb{R}$, we can form a new vector:

$$v = k_1 u_1 + k_2 u_2 + \cdots + k_m u_m$$

This vector is called a linear combination of the vectors $u_1, u_2, \ldots, u_m$.

How do you explain linear combination of vectors geometrically?

Example

1. Let $u = (2, 4, -5)$ and $v = (1, -6, 9)$, then:

$$\begin{aligned} u + v &= (2 + 1, 4 + (-6), -5 + 9) = (3, -2, 4) \\ 4u &= (8, 14, -20) \\ -v &= (-1, 6, -9) \\ 3u - 2v &= (6, 12, -15) + (-2, 12, -18) \end{aligned}$$

2. Let $u = \begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix}$ and $v = \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix}$, then:

$$2u - 3v = \begin{bmatrix} 4 \\ 6 \\ -8 \end{bmatrix} + \begin{bmatrix} -9 \\ 3 \\ 6 \end{bmatrix} = \begin{bmatrix} -5 \\ 9 \\ -2 \end{bmatrix}$$

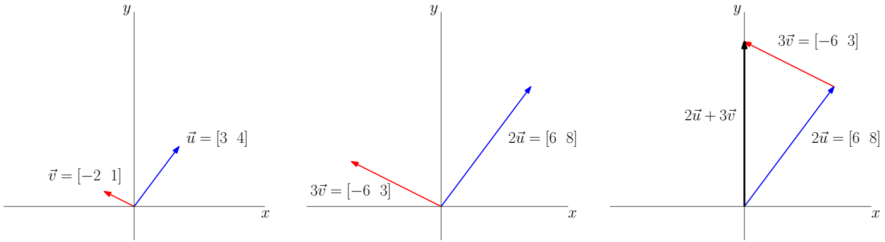
Geometric interpretation of linear combination

How would you interpret linear combination of vectors geometrically?

See it as a combination of **scaling** and **moving** vectors in a space

Example

Given a vector $\vec{u} = [3/4]$ and $\vec{v} = [-2/1]$. How do you explain $2\vec{u} + 3\vec{v}$?



Geometric interpretation of linear combination

[1 0] and [0 1] are "special vectors" in the 2D-space. Can you guess why?

Every vector u in $\mathbb{R}^2$ can be represented as a linear combination of vectors $x_1 = [1\ 0]$ and $x_2 = [0\ 1]$, i.e.:

> *For every $u \in \mathbb{R}^2$, there exist a constant $c_1, c_2 \in \mathbb{R}$ such that $u = c_1 x_1 + c_2 x_2$.*
>
> *In particular, if $u = [a_1\ \ a_2]$ then $u = a_1 x_1 + a_2 x_2$.*

Example

$[4\ 3] = 4[1\ 0] + 3[0\ 1]$

- What are the special vectors in the 3D-space?
- What about the nD-space?

Geometric interpretation of linear combination

The set

$\{x_i,\ i \in \{1, 2, \ldots, n\} \mid x_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ 1 is at the i-th position

is the set of special vectors in the n-space. (In the previous slide, we denote them by $e_1, e_2, \ldots, e_n$.)

So any vector $u = (a_1, a_2, \ldots, a_n)$ can be written as:

$$u = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n$$

We say that $\{x_1, x_2, \ldots, x_n\}$ spans $\mathbb{R}^n$.

A more formal definition will be discussed later.

Part 4: Linear Independence of Vectors

Linear independence

Given a system:

$$\begin{bmatrix} 1 & 2 & -3 \\ 3 & 5 & 9 \\ 5 & 9 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The system can be written as a vector equation:

$$x_1 \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 5 \\ 9 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 9 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The vector equation has the trivial solution:

$$x_1 = 0, x_2 = 0, x_3 = 0$$

Is there any other solution?

Linear independence

Definition (Linear independence)

A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p\} \in \mathbb{R}^n$ is said to be linearly independent if the vector equation:

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_p\mathbf{v}_p = \mathbf{0}$$

has only the trivial solution.

Definition

The set $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p\} \in \mathbb{R}^n$ is said to be linearly dependent if there exists $c_1, c_2, \dots, c_n \in \mathbb{R}^n$ which are not all 0, s.t.

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p = \mathbf{0}$$

Simply saying, two vectors are linearly independent if none of them can be expressed as a linear combination of the others.

Example of linear independence of vectors

Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 5 \\ 9 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} -3 \\ 9 \\ 3 \end{bmatrix}$.

- Determine whether $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent.

Solution:

Example of linear independence of vectors

Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 5 \\ 9 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} -3 \\ 9 \\ 3 \end{bmatrix}$.

- Determine whether $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent.

Solution:

Solve the system:

$$x_1 \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 5 \\ 9 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 9 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We can perform elementary row operations on the augmented matrix:

$$\begin{bmatrix} 1 & 2 & -3 & 0 \\ 3 & 5 & 9 & 0 \\ 5 & 9 & 3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & -1 & 18 & 0 \\ 0 & -1 & 18 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & -1 & 18 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

What can you conclude?

Example of linear dependence of vectors

Given $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 5 \\ 9 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} -3 \\ 9 \\ 3 \end{bmatrix}$. We have relation:

$$-33 \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} + 18 \begin{bmatrix} 2 \\ 5 \\ 9 \end{bmatrix} + 1 \begin{bmatrix} -3 \\ 9 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or equivalently,

$$\begin{bmatrix} 1 & 2 & -3 \\ 3 & 5 & 9 \\ 5 & 9 & 3 \end{bmatrix} \begin{bmatrix} -33 \\ 18 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Each linear dependence relation among the columns of A corresponds to a nontrivial solution to $A\mathbf{x} = \mathbf{0}$.

Exercise 1

Determine the linear independence of the following set of vectors:

1. $\{\mathbf{v}_1\} = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$
2. $\{\mathbf{u}_1, \mathbf{u}_2\} = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \end{bmatrix} \right\}$
3. $\{\mathbf{v}_1, \mathbf{v}_2\} = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right\}$

Solution:

Conclusion

How to check that a set containing one vector is linearly independent?

How to check that a set containing two vectors is linearly independent?

Conclusion

How to check that a set containing one vector is linearly independent?

Answer: $\{\mathbf{v}_1\}$ is linearly independent when $\mathbf{v}_1 \neq \mathbf{0}$

How to check that a set containing two vectors is linearly independent?

Answer:

- $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly dependent if at least one vector is a multiple of the other;
- $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent if and only if neither of the vectors is a multiple of the other.

Part 5: Numerical Computations of Vectors in $\mathbb{R}^n$

Properties of vectors under operations

Theorem

For any vectors $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ *and any scalars* $k, k' \in \mathbb{R}$,

1. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ *(associative)*
2. $\mathbf{u} + 0 = \mathbf{u}$ *(identity elt w.r.t. addition)*
3. $\mathbf{u} + (-\mathbf{u}) = 0$ *(two opposite vectors)*
4. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ *(commutative)*
5. $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$ *(distributive w.r.t. scalar mult.)*
6. $(k + k')\mathbf{u} = k\mathbf{u} + k'\mathbf{u}$
7. $(kk')\mathbf{u} = k(k'\mathbf{u})$
8. $1\mathbf{u} = \mathbf{u}$ *(identity elt w.r.t. multiplication)*

Note: Suppose $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^n$, and $\mathbf{u} = k\mathbf{v}$ for some $k \in \mathbb{R}$. Then $\mathbf{u}$ is called the multiple of $\mathbf{v}$. If $k > 0$, then $\mathbf{u}$ and $\mathbf{v}$ have the same direction, and if $k < 0$, then they are in opposite direction.

Exercise

to be continued...